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ORIGINAL PAPER

New algorithm for the numerical solution of the integro-differential equation with an integral boundary condition

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Abstract In this paper, a sequence of approximate solution converging uniformly to the exact solution for a class of integro-differential equation with an integral boundary condition arising in chemical engineering, underground water flow and population dynamics and other field of physics and mathematical chemistry is obtained by using an iterative method. Its exact solution is represented in the form of series in the reproducing kernel space. The n-term approximation $u_n(x)$ is proved to converge to the exact solution $u(x)$. Moreover, the first derivative of $u_n(x)$ is also convergent to the first derivative of $u(x)$.

Keywords Integro-differential equation · Integral boundary condition · Reproducing kernel space · Iterative method

1 Introduction

We consider the integro-differential equation with an integral boundary condition

$$\begin{cases} v'(x) = f\left(x, v(x), \int_0^1 k(s)v(s)\mathrm{d}s\right), \quad 0 \le x \le 1, \\ v(0) = \lambda v(1) + \int_0^1 D(s)v(s)\mathrm{d}s + d, \quad d \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $f \in C([0,1] \times \mathbb{R}^2, \mathbb{R})$, $k, D \in C([0,1], \mathbb{R})$ and $\lambda \in \mathbb{R}$.

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It is well known that non-local conditions came up when the values of function on the boundary were connected to the values inside the domain. They arises in a variety of different scientific fields such as chemical engineering, underground water flow and population dynamics and other field of physics and mathematical chemistry. There were extensive literatures on non-local problems and the boundary value problems (BVPs) involving integral boundary conditions had been received considerable attention. A lot of methods were applied to solve the problems such as functional method, energy method, Galerkin method and discretization method. For BVPs with integral boundary conditions and comments on their importance, we refer the reader to the papers by Gallardo [1], Karakostas and Tsamatos [2], Lomtatidze and Malaguti [3] and the references therein. In [4], the authors used the method of lower and upper solutions combined with monotone iterative techniques successfully for problems (1.1). However, the literature of numerical analysis contains little on the solution of (1.1).

So far, many classical problems such as population models and complex dynamics had been solved in reproducing kernel space [5,6]. In [7–11], many two-point BVPs were solved in reproducing kernel space which satisfied two-point boundary conditions.

In this paper, a class of integral boundary problems which satisfy integral boundary conditions in reproducing kernel space $W_2[0, 1]$ had been solved. A sequence of approximate solution converging uniformly to the exact solution of integro-differential equation with an integral boundary condition was obtained by using an iterative method.

In order to put boundary conditions of (1.1) into $W_2[0, 1]$ constructed in the following sections, we must homogenize these conditions. Through transformation of function, (1.1) can be converted into the equivalent form

$$\begin{cases} u'(x) = F(x, u(x), \int_0^1 h(s)u(s)\mathrm{d}s, & 0 \le x \le 1, \\ u(0) = \lambda u(1) + \int_0^1 H(x)u(x)\mathrm{d}x, \end{cases} \tag{1.2}$$

where $F \in C([0, 1] \times \mathbb{R}^2, \mathbb{R})$, $h, H \in C([0, 1], \mathbb{R})$ and $\lambda \in \mathbb{R}$.

We give the representation of exact solution and approximate solution of (1.2) in $W_2[0, 1]$. The advantages of this method are as follows: first, the conditions for determining solution in (1.2) can be imposed on $W_2[0, 1]$ and therefore the reproducing kernel satisfying the conditions for determining solution can be calculated. We will use the kernel to solve problems. Second, the iterative sequence $u_n(x)$ converges to the solution $u(x)$ in C^1.

The paper is organized as follows. In Sect. 2, some definitions of the reproducing kernel space and a linear operator were introduced. Section 3 provide the main results. An iterative sequence is developed for the kind of problems in $W_2[0, 1]$. Four numerical experiments shown that our methods is efficient in Sect. 4. Section 5 is the conclusions.

2 Preliminaries

In the section, some reproducing kernel spaces are introduced for solving the solution of (1.2).

2.1 The reproducing kernel space $W_2[0, 1]$

The inner product space $W_2[0, 1]$ (see [12]) is defined by

$$W_2[0,1] = \Big\{ u(x) | u'(x) \text{ is a absolutely continuous real valued function,}$$
$$u''(x) \in L^2[0,1], u(0) = \lambda u(1) + \int_0^1 H(x)u(x)\mathrm{d}x \Big\}.$$

The inner product and norm are defined respectively by

$$\langle u(x), v(x) \rangle_{W_2} = \sum_{i=0}^{1} u^{(i)}(0)v^{(i)}(0) + \int_0^1 u''(x)v''(x)\,\mathrm{d}x, \tag{2.1}$$

$$\|u\|_{W_2} = \sqrt{\langle u, u \rangle_{W_2}}, \tag{2.2}$$

where $u, v \in W_2[0, 1]$.

Theorem 2.1 *Space $W_2[0, 1]$ is a complete reproducing kernel space. That is, for each fixed $x \in [0, 1]$, there exists $K_2(y, x) \in W_2[0, 1]$, such that $\langle u(y), K_2(y, x) \rangle_{W_2} = u(x)$ for any $u(y) \in W_2[0, 1]$ and $y \in [0, 1]$. The reproducing kernel $K_2(y, x)$ can be written as*

$$K_2(y,x) = \begin{cases} \sum_{i=1}^{4} a_i(x)y^{i-1} + c_1(x)H_1(y), & y \le x, \\ \sum_{i=1}^{4} b_i(x)y^{i-1} + c_1(x)H_1(y), & y > x, \end{cases} \tag{2.3}$$

where $H_1(y) = \int_0^y \int_0^y \int_0^y \int_0^y H(y)dydydydy$.

Proof (i) The proof of the completeness and reproducing property of $W_2[0, 1]$ is similar to the proof of Theorem 1.3.1 in [12].

(ii) Now, let's find out the expression of the reproducing kernel function $K_2(y, x)$ in $W_2[0, 1]$.

Through several integration by parts for (2.1), we have

$$\begin{aligned}\langle u(y), K_2(y,x)\rangle_{W_2} &= \sum_{i=0}^{1} u^{(i)}(0)[\partial_y^i K_2(0,x) - (-1)^{1-i}\partial_y^{3-i} K_2(0,x)] \\ &\quad + \sum_{i=0}^{1} (-1)^{1-i} u^{(i)}(1)\partial_y^{3-i} K_2(1,x) \\ &\quad + \int_0^1 u(y)\partial_y^4 K_2(y,x)\,\mathrm{d}y.\end{aligned}$$

Since $u(x) \in W_2[0, 1]$, it follows that $u(0) = \lambda u(1) + \int_0^1 H(x)u(x)\mathrm{d}x$, then

$$\begin{aligned}\langle u(y), K_2(y,x)\rangle_{W_2} &= u(0)[K_2(0,x) + \partial_y^3 K_2(0,x)] \\ &\quad + u'(0)[\partial_y^1 K_2(0,x) - \partial_y^2 K_2(0,x)] \\ &\quad + u'(1)\partial_y^2 K_2(1,x) - u(1)\partial_y^3 K_2(1,x) \\ &\quad + \int_0^1 u(y)\partial_y^4 K_2(y,x)\,\mathrm{d}y \\ &\quad + c_1(x)[u(0) - \lambda u(1) - \int_0^1 H(y)u(y)\mathrm{d}y] \\ &= u(0)[K_2(0,x) + \partial_y^3 K_2(0,x) + c_1(x)] + u'(0)[\partial_y^1 K_2(0,x) \\ &\quad - \partial_y^2 K_2(0,x)] + u'(1)\partial_y^2 K_2(1,x) \\ &\quad - u(1)[\partial_y^3 K_2(1,x) + c_1(x)\lambda] \\ &\quad + \int_0^1 u(y)[\partial_y^4 K_2(y,x) - c_1(x)H(y)]\mathrm{d}y \end{aligned} \tag{2.4}$$

Note that the property of the reproducing kernel $\langle u(y), K_2(y,x)\rangle_{W_2} = u(x)$, $K_2(y,x)$ is the solution of the following generalized differential equation

$$\begin{cases} \partial_y^4 K_2(y,x) - c_1(x)H(y) = \delta(y-x), \\ K_2(0,x) + \partial_y^3 K_2(0,x) + c_1(x) = 0, \\ \partial_y^1 K_2(0,x) - \partial_y^2 K_2(0,x) = 0, \\ \partial_y^2 K_2(1,x) = 0, \\ \partial_y^3 K_2(1,x) + c_1(x)\lambda = 0. \end{cases} \tag{2.5}$$

While $y \neq x$, it is easy to know that $K_2(y,x)$ is the solution of the following constant linear homogeneous differential equation with four-order, i.e.

$$\partial_y^4 K_2(y,x) - c_1(x)H(y) = 0, \tag{2.6}$$

with the boundary conditions

$$\begin{cases} K_2(0,x) + \partial_y^3 K_2(0,x) + c_1(x) = 0, \\ \partial_y^1 K_2(0,x) - \partial_y^2 K_2(0,x) = 0, \\ \partial_y^2 K_2(1,x) = 0, \\ \partial_y^3 K_2(1,x) + c_1(x)\lambda = 0. \end{cases} \tag{2.7}$$

We know that Eq. (2.7) has the characteristic equation $\lambda^4 = 0$ and the eigenvalue $\lambda = 0$ is a root whose multiplicity is four. Therefore, the general solution of (2.7) is kernel function $K_2(y,x)$. It can be written by (2.3).

Now we are ready to calculate the coefficient $a_i(x), b_i(x)(i = 1, 2, \ldots, 4)$ and $c_1(x)$ in (2.3).

Since $\partial_y^4 K_2(y,x) - c_1 H(y) = \delta(y-x)$, we have

$$\partial_y^k K_2(x+0,x) = \partial_y^k K_2(x-0,x), \quad k = 0, 1, 2, \tag{2.8}$$

and

$$\partial_y^3 K_2(x+0,x) - \partial_y^3 K_2(x-0,x) = 1. \tag{2.9}$$

Note that $K_2(y,x) \in W_2[0,1]$, it follows that

$$K_2(0,x) = \lambda K_2(1,x) + \int_0^1 H(y)K_2(y,x)\mathrm{d}y \tag{2.10}$$

From (2.7) and (2.8)–(2.10), the unknown coefficient of (2.3) can be obtained. □

2.2 The reproducing kernel space $W_1[0,1]$

The inner product space $W_1[0,1]$ (see [13]) is defined by

$W_1[0,1] = \{u(x) \mid u \text{ is a absolutely continuous real valued function, } u' \in L^2[0,1]\}$

The inner product and norm are given respectively by

$$\langle u(x), v(x)\rangle_{W_1} = u(0)v(0) + \int_0^1 u'(x)v'(x)dx, \tag{2.11}$$

$$\|u\|_{W_1} = \sqrt{\langle u, u\rangle_{W_1}}, \tag{2.12}$$

where $u(x), v(x) \in W_1[0,1]$.

In [13], it has been proved that $W_1[0,1]$ is also a complete reproducing kernel space and its reproducing kernel is

$$K_1(y,x) = \begin{cases} 1+y, & y \le x, \\ 1+x, & y > x. \end{cases}$$

2.3 Introduction into a linear operator $\mathbb{L}$

Let $\mathbb{L}u = u'$, $\mathbb{L} : W_2[0,1] \to W_1[0,1]$, then (1.2) can be converted into the form as follows

$$\begin{cases} \mathbb{L}u = G(x,u(x),u(s)), 0 \le x \le 1, \\ u(0) = \lambda u(1) + \int\limits_0^1 H(x)u(x)\mathrm{d}x, \end{cases} \tag{2.13}$$

where $G(x,u(x),u(s)) = F(x,u(x),\int_0^1 h(s)u(s)\mathrm{d}s) \in W_2[0,1]$ and $G(x,y,z) \in W_1[0,1]$ as $x \in [0,1]$, $y(x), z(x) \in W_2[0,1]$. It is easy to prove that $\mathbb{L}$ is a bounded linear operator.

Now, we construct an orthogonal function system.

Let $\varphi_i(x) = K_1(x,x_i)$, $\psi_i(x) = \mathbb{L}^*\varphi_i(x)$, where $\mathbb{L}^*$ is the conjugate operator of $\mathbb{L}$. In terms of the properties of $K_1(y,x)$, one obtains

$$\langle u(x), \psi_i(x)\rangle_{W_2} = \langle \mathbb{L}u(x), \varphi_i(x)\rangle_{W_1} = \mathbb{L}u(x_i), \quad i = 1, 2, \ldots.$$

We collect two lemmas in [14] for future use.

Lemma 2.1 *If* $\{x_i\}_{i=1}^{\infty}$ *is dense on* $[0,1]$, *then* $\{\psi_i(x)\}_{i=1}^{\infty}$ *is a complete system of* $W_2[0,1]$ *and* $\psi_i(x) = \mathbb{L}_y K_2(y,x)|_{y=x_i}$. *The subscript* y *by the operator* $\mathbb{L}$ *indicates that the operator* $\mathbb{L}$ *applies to the function of* y.

Lemma 2.2 *If* $u(x) \in W_2[0,1]$, *then there exists* $M_1 > 0$, *such that* $\|u\|_{C^1[0,1]} \le M_1\|u\|_{W_2}$, *where* $\|u\|_{C^1[0,1]} = \max\limits_{x\in[0,1]} |u(x)| + \max\limits_{x\in[0,1]} |u'(x)|$.

The orthonormal system $\{\overline{\psi}_i(x)\}_{i=1}^{\infty}$ of $W_2[0,1]$ can be derived from Gram-Schmidt orthogonalization process of $\{\psi_i(x)\}_{i=1}^{\infty}$,

$$\overline{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x), \tag{2.14}$$

where β_{ik} are orthogonalization coefficients.

Hence $\forall u(x) \in W_2[0,1]$, $u(x)$ can be expanded in terms of Fourier series about normal orthogonal system

$$u(x) = \sum_{i=1}^{\infty} \langle u(x), \overline{\psi}_i(x)\rangle_{W_2} \overline{\psi}_i(x). \tag{2.15}$$

3 An iterative method for the approximate solution of (2.13)

Theorem 3.1 *If* $\{x_i\}_{i=1}^{\infty}$ *is dense on* $[0, 1]$ *and* $u(x) \in W_2[0, 1]$ *is the solution of* (2.13)*, then* $u(x)$ *satisfies the form*

$$u(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} G(x_k, u(x_k), u(s))\bar{\psi}_i(x), \tag{3.1}$$

Proof Since $u(x) \in W_2[0, 1]$, due to $W_2[0, 1]$ is the Hilbert space, the series $\sum\limits_{i=1}^{\infty}\langle u(x), \overline{\psi}_i(x)\rangle_{W_2}\overline{\psi}_i(x)$ is convergent in the norm of $\|\cdot\|_{W_2}$. Note that $\langle v(x), \varphi_i(x)\rangle_{W_1} = v(x_i)$ for each $v(x) \in W_1[0, 1]$.

By Lemma 2.1, (2.14) and (2.15) one obtain

$$\begin{aligned} u(x) &= \sum_{i=1}^{\infty}\langle u(x), \bar{\psi}_i(x)\rangle_{W_2}\bar{\psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik}\langle u(x), \psi_k(x)\rangle_{W_2}\bar{\psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik}\langle u(x), \mathbb{L}^*\varphi_k(x)\rangle_{W_2}\bar{\psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik}\langle \mathbb{L}u(x), \varphi_k(x)\rangle_{W_1}\bar{\psi}_i(x). \end{aligned}$$

If $u(x) \in W_2[0, 1]$ is the exact solution of (2.13), then $\mathbb{L}u = G(x, u(x), u(s))$, hence we have

$$\begin{aligned} u(x) &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik}\langle G(x, u(x), u(s)), \varphi_k(x)\rangle_{W_1}\bar{\psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} G(x_k, u(x_k), u(s))\bar{\psi}_i(x). \end{aligned}$$

□

Remark Case (i): If (2.13) is linear, that is, $G(x, u(x), u(s)) = G(x)$, then the analytical solution of (2.13) can be obtained directly by (3.1).

Case (ii): If (2.13) is nonlinear, that is, $G(x, u(x), u(s))$ depends on u, then the solution of (2.13) can be obtained by the following iterative method.

We construct an iterative sequence $u_n(x)$, putting

$$\begin{cases} \text{any fixed } u_0(x) \in W_2[0, 1], \\ u_n(x) = \sum\limits_{i=1}^{n} A_i\bar{\psi}_i(x), \end{cases} \tag{3.2}$$

where

$$\begin{cases} A_1 = \beta_{11} G(x_1, u_0(x_1), u_0(s)), \\ A_2 = \sum_{k=1}^{2} \beta_{2k} G(x_k, u_{k-1}(x_k), u_{k-1}(s)), \\ \cdots \\ A_n = \sum_{k=1}^{n} \beta_{nk} G(x_k, u_{k-1}(x_k), u_{k-1}(s)). \end{cases} \tag{3.3}$$

Next, we proof $u_n(x)$ in iterative formula (3.2) is convergent to the exact solution of (2.13).

Theorem 3.2 *Suppose the following conditions are satisfied: (i)* $\|u_n\|_{W_2}$ *is bounded; (ii)* $\{x_i\}_{i=1}^{\infty}$ *is dense in* $[0, 1]$*; (iii)* $G(x, y(x), z(x)) \in W_1[0, 1]$ *for any* $y(x), z(x) \in W_2[0, 1]$*. Then* $u_n(x)$ *in iterative formula* (3.2) *converges to the exact solution* $u(x)$ *of (2.13) in* $W_2[0, 1]$ *and*

$$u(x) = \sum_{i=1}^{\infty} A_i \bar{\psi}_i,$$

where A_i *are given by* (3.3).

Proof (i) First, we will prove the convergence of $u_n(x)$.
By (3.2), we have

$$u_{n+1}(x) = u_n(x) + A_{n+1}\bar{\psi}_{n+1}(x), \tag{3.4}$$

then the orthonormality of $\{\bar{\psi}(x)\}_{i=1}^{\infty}$ yields

$$\|u_{n+1}\|_{W_2}^2 = \|u_n\|_{W_2}^2 + (A_{n+1})^2 = \cdots = \sum_{i=1}^{n+1} (A_i)^2. \tag{3.5}$$

From boundedness of $\|u_n\|_{W_3}$, we have $\sum_{i=1}^{\infty}(A_i)^2 < \infty$, i.e. $\{A_i\} \in l^2(i = 1, 2, \cdots)$.
Let $m > n$, in view of $(u_m - u_{m-1}) \perp (u_{m-1} - u_{m-2}) \perp \cdots \perp (u_{n+1} - u_n)$, it follows that

$$\begin{aligned} \|u_m - u_n\|_{W_3}^2 &= \|u_m - u_{m-1} + u_{m-1} - u_{m-2} + \cdots + u_{n+1} - u_n\|_{W_3}^2 \\ &= \|u_m - u_{m-1}\|_{W_3}^2 + \cdots + \|u_{n+1} - u_n\|_{W_3}^2 \\ &= \sum_{i=n+1}^{m} (A_i)^2 \to 0, \ (m, n \to \infty). \end{aligned} \tag{3.6}$$

Considering the completeness of $W_2[0, 1]$, there exists $u(x) \in W_2[0, 1]$, such that

$$u_n(x) \xrightarrow{\|\cdot\|_{W_2}} u(x), \quad \text{as } n \to \infty$$

(ii) Second, we will prove $u(x)$ is the solution of (2.13).

By Lemma 2.2 and (i) of Theorem 3.2, we know $u_n(x)$ converge uniformly to $u(x)$. It follows that, on taking limits in (3.2),

$$u(x) = \sum_{i=1}^{\infty} A_i \bar{\psi}_i.$$

Note that

$$\begin{aligned}(Lu)(x_j) &= \sum_{i=1}^{\infty} A_i \langle \mathbb{L}\bar{\psi}_i(x), \varphi_j(x)\rangle_{W_1} = \sum_{i=1}^{\infty} A_i \langle \bar{\psi}_i(x), \mathbb{L}^*\varphi_j(x)\rangle_{W_2} \\ &= \sum_{i=1}^{\infty} A_i \langle \bar{\psi}_i(x), \psi_j(x)\rangle_{W_2}.\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{j=1}^{n} \beta_{nj}(Lu)(x_j) &= \sum_{i=1}^{\infty} A_i \left\langle \bar{\psi}_i(x), \sum_{j=1}^{n} \beta_{nj}\psi_j(x) \right\rangle_{W_2} \\ &= \sum_{i=1}^{\infty} A_i \langle \bar{\psi}_i(x), \bar{\psi}_n(x)\rangle_{W_2} = A_n.\end{aligned}$$

If $n = 1$, then

$$(\mathbb{L}u)(x_1) = G(x_1, u_0(x_1), u_0(s)). \tag{3.7}$$

If $n = 2$, then

$$\begin{aligned}\beta_{21}(\mathbb{L}u)(x_1) + \beta_{22}(\mathbb{L}u)(x_2) = \beta_{21}G(x_1, u_0(x_1), u_0(s)) \\ + \beta_{22}G(x_2, u_1(x_2), u_1(s)).\end{aligned} \tag{3.8}$$

It is clear that

$$(\mathbb{L}u)(x_2) = G(x_2, u_1(x_2), u_1(s)).$$

Furthermore, it is easy to see by induction that

$$(\mathbb{L}u)(x_j) = G(x_j, u_{j-1}(x_j), u_{j-1}(s)). \tag{3.9}$$

Since $\{x_i\}_{i=1}^{\infty}$ is dense on interval $[0, 1]$, for any $y \in [0, 1]$, there exists subsequence $\{x_{n_j}\}$, such that

$$x_{n_j} \to y, \quad \text{as } j \to \infty.$$

Hence, let $j \to \infty$ in (3.9), by the convergence of $u_n(x)$ and Lemma 2.3, we have

$$(\mathbb{L}u)(y) = G(y, u(y), u(s)), \tag{3.10}$$

that is, $u(x)$ is the solution of (2.13) and

$$u(x) = \sum_{i=1}^{\infty} A_i \bar{\psi}_i, \tag{3.11}$$

where A_i are given by (3.3). □

Theorem 3.3 *If* $\|u_n - u\|_{W_2} \to 0$, $x_n \to x$, $(n \to \infty)$ *and* $G(x, y, z)$ *is continuous with respect to* x, y, z *for* $x \in [0, 1]$, $y, z \in (-\infty, +\infty)$ *, then*

$$G(x_n, u_{n-1}(x_n), u_{n-1}(s)) \to G(x, u(x), u(s)) \text{ as } n \to \infty.$$

Proof Since $\|u_n - u\|_{W_2} \to 0$, $(n \to \infty)$, by Lemma 2.2, we know $u_n(x)$ is convergent uniformly to $u(x)$. □

From Lemma 2.2, we have the following the corollary.

Corollary 3.1 *Assume that the conditions of Theorem 3.2 hold, then* $u_n(x)$ *in 3.2 satisfies* $\|u_n - u\|_{C^1[0,1]} \to 0$, $n \to \infty$, *where* $u(x)$ *is the solution of (2.13).*

4 Numerical experiments

Mathematical modeling of real-life, physics and engineering problems usually results in functional equations, e.g. partial differential equations, integral and integro-differential equations, stochastic equations and others. Many mathematical formulation of physical phenomena contain integro-differential equations, these equations arise in fluid dynamics, biological models and chemical kinetics. Integro-differential equations are usually difficult to solve analytically so it is required to obtain an efficient approximate solution. In order to test the utility of the proposed method, we have solved the following four problems. All computations are performed by the Mathematica 5.0 software package.

Problem 1 Consider the following problem

$$\begin{cases} u'(x) - u^2(x) + \int_0^1 u(s)\mathrm{d}s = f(x), \quad 0 \le x \le 1, \\ u(0) = \int_0^1 u(x)\mathrm{d}x. \end{cases}$$

where $f(x) = 18(-3+e) + 9e^x x - (9 + 9e^x(-1+x) + 18(-3+e)x)^2$. The exact solution is $u(x) = 9(x-1)e^x + 9(2e-6)x + 9$. Numerical results are displayed in Table 1 and Fig. 1. The root-mean-square errors for the first derivative is 5.52461E-3.

Table 1 The numerical results for Problem 1

Node	True solution $u(x)$	Approximate solution $u_{100}(x)$	Absolute error	Relative error
0.1	−0.4589770	−0.4590070	3.01330E-5	6.56525E-5
0.2	−0.8082850	−0.8083340	4.90581E-5	6.06940E-5
0.3	−1.0253900	−1.0254400	5.56755E-5	5.42970E-5
0.4	−1.0842200	−1.0842800	5.16563E-5	4.76436E-5
0.5	−0.9547090	−0.9547490	4.00630E-5	4.19635E-5
0.6	−0.6021840	−0.6022070	2.32934E-5	3.86815E-5
0.7	0.0132187	0.0132175	1.18604E-6	8.97239E-5
0.8	0.9372850	0.9373160	3.17482E-5	3.38725E-5
0.9	2.2225200	2.2226200	9.85777E-5	4.43539E-5
1	3.9290700	3.9292900	2.13947E-4	5.44523E-5

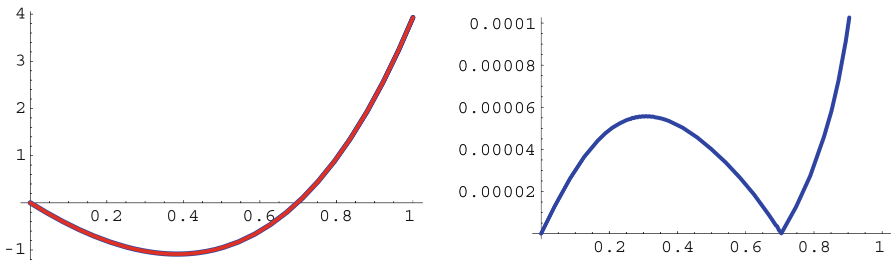

Fig. 1 The superimposed image of $u(x)$ with $u_{100}(x)$ and the $|u(x) - u_{100}(x)|$ for Problem 1

Problem 2 Consider the following problem

$$\begin{cases} u'(x) + \frac{u(x)}{1+\int_0^1 u(s)\mathrm{d}s} = \frac{-3-4x}{26}, 0 < x < 1, \\ u(0) - \frac{1}{3}u(1) = \int_0^1 u(x)\mathrm{d}x, \end{cases}$$

The exact solution is $u(x) = 1 + \frac{x}{4}$. Numerical results are displayed in Table 2 and Fig. 2. The root-mean-square errors for the first derivative is 5.95426E-5.

Problem 3 Consider the linear boundary value problem for the integro-differential equation which arises in chemical kinetics (see Ref. [15])

$$\begin{cases} y^{(4)}(x) = x(1+e^x) + 3e^x + y(x) - \int_0^x y(t)\mathrm{d}t, \quad 0 < x < 1, \\ y(0) = 1, \quad y'(0) = 1, \\ y(1) = 1 + e, \quad y'(1) = 2e. \end{cases}$$

The exact solution is $y(x) = 1 + xe^x$. Numerical results are displayed in Table 3 and Fig. 3. The root-mean-square (RMS) errors for the derivatives are showed in Table 4.

Table 2 The numerical results for Problem 2

Node	True solution $u(x)$	Approximate solution $u_{100}(x)$	Absolute error	Relative error
0.1	2.05	2.04947	5.25709E-4	2.56443E-4
0.2	2.10	2.09946	5.39337E-4	2.56827E-4
0.3	2.15	2.14945	5.52917E-4	2.57170E-4
0.4	2.20	2.19943	5.66347E-4	2.57431E-4
0.5	2.25	2.24942	5.79521E-4	2.57565E-4
0.6	2.30	2.29941	5.92329E-4	2.57530E-4
0.7	2.35	2.34940	6.04659E-4	2.57302E-4
0.8	2.40	2.39938	6.16394E-4	2.56831E-4
0.9	2.45	2.44937	6.27415E-4	2.56088E-4
1.0	2.50	2.49936	6.37599E-4	2.55040E-4

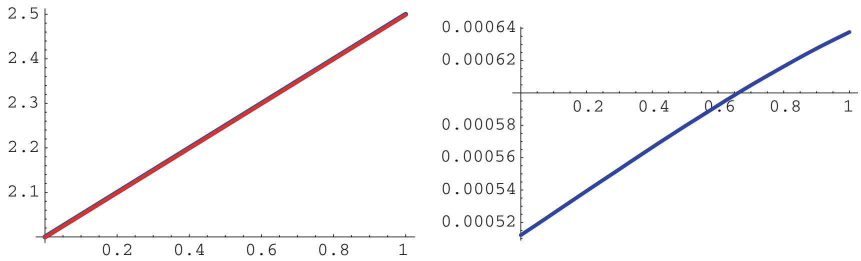

Fig. 2 The superimposed image of $u(x)$ with $u_{100}(x)$ and the $|u(x) - u_{100}(x)|$ for Problem 2

Table 3 The numerical results for Problem 3

Node	True solution $y(x)$	Approximate solution $y_{100}(x)$	Absolute error	Relative error
0.1	1.11052	1.11052	2.34417E-8	2.11088E-8
0.2	1.24428	1.24428	7.67790E-8	6.17055E-8
0.3	1.40496	1.40496	1.37083E-7	9.75710E-8
0.4	1.59673	1.59673	1.85564E-7	1.16215E-7
0.5	1.82436	1.82436	2.08645E-7	1.14366E-7
0.6	2.09327	2.09327	1.99202E-7	9.51631E-8
0.7	2.40963	2.40963	1.57945E-7	6.55477E-8
0.8	2.78043	2.78043	9.49140E-8	3.41364E-8
0.9	3.21364	3.21364	3.10742E-8	9.66947E-9
1.0	3.71828	3.71828	0	0

Problem 4 Consider the nonlinear boundary value problem for the integro-differential equation which arises in chemical kinetics (see Ref. [15])

$$\begin{cases} y^{(4)}(x) = 1 + \int_0^x e^{-t} y^2(t)\mathrm{d}t, \quad 0 < x < 1, \\ y(0) = 1, \quad y'(0) = 1, \\ y(1) = e, \quad y'(1) = e. \end{cases}$$

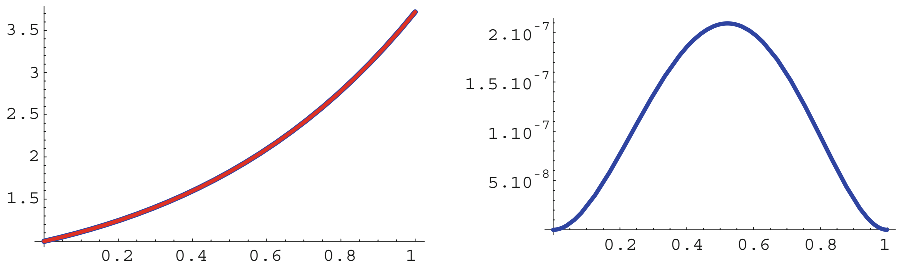

Fig. 3 The superimposed image of $y(x)$ with $y_{100}(x)$ and the $|y(x) - y_{100}(x)|$ for Problem 3

Table 4 The RMS errors for the partial derivatives for example 3

$\sqrt{\frac{\sum_{i=1}^{10}[y'(0.1i,0.1i)-y'_{100}(0.1i,0.1i)]^2}{10}}$	4.62606E-7	$\sqrt{\frac{\sum_{i=1}^{10}[y''(0.1i,0.1i)-y''_{100}(0.1i,0.1i)]^2}{10}}$	3.42337E-6
$\sqrt{\frac{\sum_{i=1}^{10}[y'''(0.1i,0.1i)-y'''_{100}(0.1i,0.1i)]^2}{10}}$	2.66790E-5	$\sqrt{\frac{\sum_{i=1}^{10}[y''''(0.1i,0.1i)-y''''_{100}(0.1i,0.1i)]^2}{10}}$	9.27720E-5

Table 5 The numerical results for Problem 4

Node	True solution $y(x)$	Approximate solution $y_{100}(x)$	Absolute error	Relative error
0.1	1.10517	1.10517	3.21511E-9	2.90916E-9
0.2	1.22140	1.22140	1.05891E-8	8.66959E-9
0.3	1.34986	1.34986	1.90069E-8	1.40807E-8
0.4	1.49182	1.49182	2.58585E-8	1.73335E-8
0.5	1.64872	1.64872	2.92105E-8	1.77171E-8
0.6	1.82212	1.82212	2.80067E-8	1.53704E-8
0.7	2.01375	2.01375	2.22897E-8	1.10687E-8
0.8	2.22554	2.22554	1.34379E-8	6.03803E-9
0.9	2.45960	2.45960	4.41125E-9	1.79348E-9
1.0	2.71828	2.71828	0	0

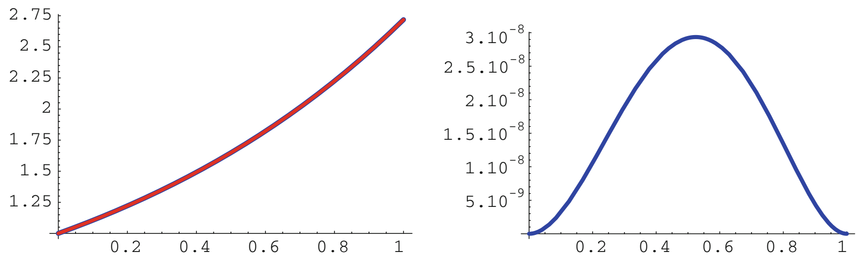

Fig. 4 The superimposed image of $y(x)$ with $y_{100}(x)$ and the $|y(x) - y_{100}(x)|$ for Problem 4

The exact solution is $y(x) = e^x$. Numerical results are displayed in Table 5 and Fig. 4. The root-mean-square (RMS) errors for the derivatives are showed in Table 6.

Table 6 The RMS errors for the partial derivatives for example 4

$\sqrt{\frac{\sum_{i=1}^{10}[y'(0.1i,0.1i)-y'_{100}(0.1i,0.1i)]^2}{10}}$	6.48532E-8	$\sqrt{\frac{\sum_{i=1}^{10}[y''(0.1i,0.1i)-y''_{100}(0.1i,0.1i)]^2}{10}}$	4.84126E-7
$\sqrt{\frac{\sum_{i=1}^{10}[y'''(0.1i,0.1i)-y'''_{100}(0.1i,0.1i)]^2}{10}}$	3.78436E-6	$\sqrt{\frac{\sum_{i=1}^{10}[y''''(0.1i,0.1i)-y''''_{100}(0.1i,0.1i)]^2}{10}}$	1.31565E-5

5 Conclusions

In summary, we use an iterative method to find the approximate solution of integro-differential equation with an integral boundary condition in the reproducing kernel space. Using the method, a sequence which is proved to converge to the exact solution uniformly is obtained. Numerical results are verified that the method employed in the paper is valid. It is worthy to note that this method can be used as a very accurate algorithm for solving linear and nonlinear integro-differential equations with integral boundary conditions arising in chemical kinetics, physics and other field of applied mathematics.

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